ALGEBRAIC CURVES
EXERCISE SHEET 4

Unless otherwise specified, k is an algebraically closed field.

Exercise 4.1. Show that all local rings of the affine line $\mathbb{A}^1_k$ are isomorphic to the same ring R.

Solution 1. Let $p \in \mathbb{A}^1_k$. It corresponds to a maximal ideal $\mathfrak{p} = (x-a)$, $a \in k$ of $k[x]$. $S = k[x] \setminus \mathfrak{p}$ is a multiplicative set. The local ring at p is $k[x]_{(x-a)}$. But there is a ring isomorphism $k[x] \to k[x]$, $x \mapsto x + a$ which sends $(x-a)$ to (x) and thus, $k[x]_{(x-a)} \simeq k[x]_{(x)} =: R$.

Exercise 4.2. An *affine algebraic group* is an affine variety G, whose underlying set is a group, such that the morphisms $i : G \to G,\ g \mapsto g^{-1}$ and $m : G \times G \to G,\ (g,h) \mapsto gh$ are polynomial maps. Let $V_1 = \mathbb{A}^1_k - \{0\}$ and $V_2 = V(xy-1)$. From the first exercise, we call R the local ring of $\mathbb{A}^1_k$ at any point.

(1) Show that $\mathcal{O}(V_1) = k[x, x^{-1}] = k[x,y]/(xy-1)$.
(2) Construct a morphism $V_2 \to \mathbb{A}^1_k$ whose image is V_1.
(3) Show that the local ring of V_2 at any point is isomorphic to R. Are V_2 and $\mathbb{A}^1_k$ isomorphic?
(4) Show that V_2 can be endowed with a structure of affine algebraic group.

Solution 2.

(1) By definition,

$$\mathcal{O}(\mathbb{A}^1_k - \{0\}) = \cap_{a \neq 0} \Big\{ \frac{f}{g} \mid g(a) \neq 0 \Big\} = k[x, \frac{1}{x}]$$

The following ring morphism has kernel $(xy-1)$ and we conclude using the isomorphism theorem.

$$\begin{aligned} k[x,y] &\to k[x, x^{-1}] \\ x &\mapsto x \\ y &\mapsto x^{-1} \end{aligned}$$

(2) The projection $(v_1, v_2) \longmapsto v_1$ works. On structure rings, it is

$$\begin{aligned} k[x] &\to k[x,y]/(xy-1) \\ x &\mapsto x \end{aligned}$$

(3) $k[x, x^{-1}]$ and $k[x]$ are not isomorphic so V_2 and $\mathbb{A}^1_k$ are not isomorphic. However, all its local rings are the local rings of V_1 which in turns are

all isomorphic to R (Check $\mathfrak{m}_{(v_1,v_2)}/\mathfrak{m}^2_{(v_1,v_2)} = (\overline{x} - v_1)$ and the fact that $\mathcal{O}_P(V) \simeq \mathcal{O}_P(\overline{V})$).

(4) V_2 has a multiplication map given by

$$m \ : \ (a,b)\cdot(c,d) \mapsto (ac,bd)$$

$(ac,bd) \in V_2$ because $acbd = (ab)(cd) = 1$. The inverse map is :

$$i \ : (a,b) \mapsto (b,a)$$

$m((a,b),(b,a)) = (ab,ba) = (1,1)$ which is the neutral element for m. This defines a structure of affine algebraic group on V_2.

Exercise 4.3. Let $V = V(y^2 - x^3)$. Let $\varphi : \mathbb{A}^1_k \to V$ be the morphism defined by $\varphi(t) = (t^2, t^3)$. From the first exercise, we call R the local ring of $\mathbb{A}^1_k$ at any point.

(1) Show φ is a bijective morphism, but is not an isomorphism.
(2) Let $P \in V$. Is the local ring of V at P isomorphic to R?

Solution 3. (1) ϕ is a bijection : there is an inverse $(a,b) \mapsto \frac{b}{a}$ on $V \setminus (0,0)$ and $\{0\} \mapsto \{(0,0)\}$ is clearly a bijection. However it is not an isomorphism, because the inverse $(a,b) \mapsto \frac{b}{a}$ does not extend to $\{0,0\}$.

We can also see it on the rings of functions, where ϕ is induced by the morphism

$$\begin{array}{rl} k[x,y] & \to k[x] \\ x & \mapsto x^2 \\ y & \mapsto x^3 \end{array}$$

The kernel is $(y^2 - x^3)$, but it is clearly not surjective because x is not in the image.

(2) There is an isomorphism $V_1 \simeq V \setminus \{(0,0)\}$ so the local rings at $p \in V \setminus \{(0,0)\}$ are isomorphic to R. It is not the case at $P = (0,0)$. Let $\mathcal{O}_{V,P}$ be the local ring at P. We can consider $\mathfrak{m}/\mathfrak{m}^2$ where $\mathfrak{m}$ is the maximal ideal of $\mathcal{O}_{V,P}$. It is a $k-$vector space and there is a basis given by x and y, so it is two-dimensional. We can compare it with $\mathfrak{m}_R/\mathfrak{m}_R^2$ where $\mathfrak{m}_R$ is the maximal ideal of R. It is clearly $1-$dimensional. $\mathfrak{m}/\mathfrak{m}^2$ is an invariant of local rings (called the Zariski cotangent space), so this shows that $\mathcal{O}_{V,P}$ and R are not isomorphic.

Exercise 4.4. Let $V = V(Y^2 - X^2(X+1))$ and x, y the residues of X, Y in $\Gamma(V)$. Let $z = \frac{y}{x} \in k(V)$. Find the poles of z and z^2.

Solution 4. Note that z is represented by $\frac{Y}{X}$ or $\frac{X(X+1)}{Y}$ in $k[X,Y]$. It has a pole at $(0,0)$. $z^2 = \frac{y^2}{x^2} = \frac{x^2(x+1)}{x^2} = x + 1$ is polynomial, it has no poles.

Exercise 4.5. Let V be an affine variety and $f \in k(V)$ a rational function. Show that f defines a continous function $U \to k$, for some non empty open subset $U \subset V$. Furthermore f is uniquely determined by this function.

Solution 5. V is irreducible so $k[V]$ is integral and we can write f as g/h with $g, h \in k[V]$. Then the zero set of h is a closed subset of V and we can take U to be its complement. The only Zariski closed subsets of k are $\varnothing, k$ and finite sets of points. Checking the continuity on singletons is enough. Using translations, it suffices to check at 0. Now $f^{-1}(0)$ is Zariski closed since $f^{-1}(0) = g^{-1}(0)$ is Zariski closed.

Using projective space : We can see f as a function $V \mapsto \mathbb{P}^1$. Then, $f^{-1}(\infty)$ is closed and its complement is the open subset U.

An open in an irreducible set is dense. So just by continuity $f_{|U}$ determines f.

Exercise 4.6. * Let $F \in k[x,y]$ be an irreducible polynomial of degree at most 2. Show that $V(F)$ is either isomorphic to $V_1 = \mathbb{A}^1_k$ or $V_2 = V(xy-1)$. Specify in which case it is isomorphic to V_1 (resp. V_2). (Hint: Use linear changes of coordinates to eliminate monomials in F)

Solution 6. A degree 1 irreducible polynomial is of the form $F = ax + by + c$ with a or $b \neq 0$. Assume $a \neq 0$. Then we have the following surjective morphism

$$\begin{aligned} k[x,y] &\to k[x] \\ x &\mapsto -a^{-1}(bx+c) \\ y &\mapsto x \end{aligned}$$

whose kernel is (F). Thus $V(F)$ is isomorphic to $\mathbb{A}^1_k$.

Now suppose F is an irreducible polynomial of degree 2 in $k[x,y]$.

We can write

$$F(x,y) = ax^2 + by^2 + cxy + dx + ey + f = 0$$

- if $a = 0$ and $b = 0$, then $c \neq 0$. Using

  $$cxy + dx + ey = c(x + \frac{e}{c})(y + \frac{d}{c}) - ed$$

  we get $F = cXY + f'$ with $X = x + \frac{e}{c}$, $Y = y + \frac{d}{c}$ and $f' = f - ed$. Then F irreducible implies $f' \neq 0$. If we write $X' = \frac{c}{f'}$, then $F = f'XY - f'$. It is then clear that $V(F) = V(XY - 1)$.

  Note that these affine changes of variables are admitted because they induce isomorphism of rings.
- if $a \neq 0$, $b = 0$, use Gauss reduction to eliminate xy, and another affine change of variable to eliminate x to get

  $$F = X^2 + e'Y + f'$$

  Then F irreducible implies $e' \neq 0$. Changing $Y' = d'Y + f'$, we get that $V(F) = V(X^2 + Y)$. But then use the isomorphism

  $$\begin{aligned} k[x,y]/(x^2 - y) &\to k[x] \\ x &\mapsto x \\ y &\mapsto x^2 \end{aligned}$$

  to get that $V(F) = V_1$
- if $a \neq 0$, $b \neq 0$, use again Gauss reduction to eliminate xy and affine transformations to eliminate linear terms. We get $F = X^2 + y^2 + f'$.

But $ax^2 + by^2$ is always reducible over an algebraically closed field, as

$$ax^2 + by^2 = (\sqrt{a}x + i\sqrt{b}y)(\sqrt{a}x - i\sqrt{b}y)$$

We can then use affine transformation in x and y to get $F = f''(XY - 1)$, so $V(F) = V_2$.